estimates on the components of the metric tensor

aim: to get bounds on the components of the metric tensor from bounds on the curvature and the injectivity radius.

Definition 1: A coordinate chart $(x^1, \cdots, x^n)$ on a Rie. manifold (M^n, g) is called harmonic if $\Delta_g x^k = 0$ for all $k = 1, 2, \cdots, n$. Since $\Delta_g x^k = g^{ij}\Gamma_{ij}^k$, we get that a coordinate chart $(x^1, \cdots, x^n)$ is harmonic if and only if for any $k = 1, 2, \cdots, n$, $g^{ij}\Gamma_{ij}^k = 0$

Remark: in harmonic coordinates, $Ric_{ij} = -\frac{1}{2} g^{mk}\frac{\partial^2 g_{ij}}{\partial x^m \partial x^k} + \cdots$

where the dots indicate lower-order terms involving at most one derivative of the metric.

proof. $Ric_{ij} = g^{mk} Rim_{jk} = -g^{mk}\frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^m \partial x^k} + \frac{\partial^2 g_{mk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ik}}{\partial x^m \partial x^j} - \frac{\partial^2 g_{mj}}{\partial x^i \partial x^k}\right)$

$g^{ij}\Gamma_{ij}^k = 0 \Rightarrow g^{ij} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l}) = 0$

$\Rightarrow g^{ij} g^{kl}(2 g_{il,j} - g_{ij,l}) = 0$

$\overset{i\to m}{\underset{对x^i求导}{\Rightarrow}} \quad g^{mj} g^{kl}\left(2\frac{\partial^2 g_{ml}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{mj}}{\partial x^i \partial x^l}\right) + \cdots = 0$

乘g_{rk} $\Rightarrow \quad g^{mj}\frac{\partial^2 g_{mr}}{\partial x^i \partial x^j} = \frac{1}{2} g^{mj}\frac{\partial^2 g_{mj}}{\partial x^i \partial x^r} + \cdots$

i.e. 有 $\quad g^{mk}\frac{\partial^2 g_{mj}}{\partial x^k \partial x^i} = \frac{1}{2} g^{mk}\frac{\partial^2 g_{mk}}{\partial x^i \partial x^j} + \cdots$

$$g^{mk}\frac{\partial^2 g_{ik}}{\partial x^m \partial x^j} = \frac{1}{2} g^{mk}\frac{\partial^2 g_{mk}}{\partial x^i \partial x^j} + \cdots$$

$\Rightarrow Ric_{ij} = -\frac{1}{2} g^{mk}\frac{\partial^2 g_{ij}}{\partial x^m \partial x^k} + \cdots$ #

Definition 2. Let (M^n, g) be a Rie. manifold, and let $x \in M$. Given $Q > 1$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$, we define the $C^{k,\alpha}$ harmonic radius at x as the largest number $r_H = r_H(Q, k, \alpha)(x)$ such that on the geodesic ball $B_x(r_H)$ of center x and radius r_H, there is a harmonic coordinate chart such that the metric tensor is $C^{k,\alpha}$ controlled in these coordinates, then

1) $Q^{-1}\delta_{ij} \le g_{ij} \le Q\,\delta_{ij}$ as bilinear forms

2) $\sum\limits_{1\le|\beta|\le k} r_H^{|\beta|}\sup\limits_{y}|\partial_\beta g_{ij}(y)| + \sum\limits_{|\beta| = k} r_H^{k+\alpha}\sup\limits_{y\neq z}\frac{|\partial_\beta g_{ij}(y) - \partial_\beta g_{ij}(z)|}{d_g(y,z)^\alpha} \le Q^{-1}$

The harmonic radius $r_H(Q, k, \alpha)(M)$ of (M, g) is now defined by
$r_H > 0$?

$r_H(Q, k, \alpha)(M) = \inf\limits_{x\in M} r_H(Q, k, \alpha)(x)$.

Problem 1.

It is easy to prove that for any $x \in M$, there is a neighborhood of x in which harmonic coordinate exist.

FACT: there always exists a smooth solution of $\Delta_g u = 0$ with $u(x) = 0$ and $\partial_i u(x)$ prescribed.

locally $\begin{cases} \Delta_g u = g^{ij} u_{ij} - g^{ij} \Gamma_{ij}^k u_k = 0 \\ u(x) = 0. \quad \partial_i u(x) = a_i . \end{cases}$ $\quad u_{ij} = \dfrac{\partial^2 u}{\partial x^i \partial x^j}$, $\quad u_k = \dfrac{\partial u}{\partial x^k}$ $\quad$ by FACT $\Rightarrow$ harmonic coor. exist!

THEOREM 1. *Let* **L** *be an elliptic operator of order m with Hölder continuous coefficients and f a Hölder continuous function. In a sufficiently small neighborhood of a point, say of the origin* $x = 0$, *there exists a solution u of the equation* $\mathbf{L}u = f$ *having the following properties:* (i) *u has Hölder continuous derivatives of order m.* (ii) *u and its derivatives up to order* $m - 1$ *have at the origin prescribed values.* (This solution is, of course, not unique.)

不要问我证明细节!
我没读. 我不关心.
我只关心这个事实.

[ref]: Partial_Differential_Equations_by_Lipman_Bers,_Fri_1170083_(z-lib.org)

Remark: One easily checks that the function $x \mapsto r_H(Q, k, \alpha)(x)$ is 1-Lipschitzian on M, since by definition. for any $x, y \in M$, $r_H(Q,k,\alpha)(y) \geq r_H(Q,k,\alpha)(x) - d_g(x,y)$.

Theorem Let $\alpha \in (0,1)$, $Q>1$, $\delta>0$ Let (M^n, g) be a Rie. manifold. and Ω an open subset of M. Set $\Omega(\delta) = \{x \in M : d_g(x, \Omega) < \delta\}$

Suppose that for some $\lambda \in \mathbb{R}$ and $i>0$, we have that for all $x \in \Omega(\delta)$

$$Ric_{(M,g)}(x) \geq \lambda g_x \quad \text{and} \quad inj_{(M,g)}(x) \geq i$$

Then there exists a positive constant $C \sim n, Q, \alpha, \delta, i, \lambda$ s.t $\sup\limits_{\Omega} r_H(Q, 0, \alpha) \geq C$.

In addition, if instead of the bound $Ric_{(M,g)}(x) \geq \lambda g_x$, we assume that for some $k \in \mathbb{N}$ and some positive constants $C(j)$, $|D^j Ric_{(M,g)}(x)| \leq C(j)$ for $j=0, \cdots, k$ and all $x \in \Omega(\delta)$, then, there exists a positive constant $C \sim n, Q, k, \alpha, \delta, i, C(j)_{0 \leq j \leq k}$, s.t. $\sup\limits_{\Omega} r_H(Q, k+1, \alpha) \geq C$.

From local analysis to global analysis

aim: to prove a packing Lemma that will be used many times

Theorem (体积比较定理)

Let (M^n, g) be a complete Rie. manifold whose Ricci curvature satisfies $Ric_{(M,g)} \geq (n-1)kg$ for some $k \in \mathbb{R}$. Then for any $0<r<R$ and any $x \in M$

$$\frac{Vol_g(B_x(R))}{Vol_g(B_x(r))} \leq \frac{V_k(R)}{V_k(r)}$$

In particular, for any $r>0$ and any $x \in M$, $Vol_g(B_x(R)) \leq V_k(r)$.

Where $V_k(t)$ denotes the volume of a ball of radius t in the complete simply connected Riemannian n-manifold of constant curvature k.

Remark: $b_n := Vol(B_1^e(0))$

FACT: $V_0(t) = b_n t^n$

$$V_k(t) = n b_n (\sqrt{k})^{-n+1} \int_0^t (\sin \sqrt{k} \rho)^{n-1} d\rho \qquad k>0$$

$$V_k(t) = n b_n (\sqrt{-k})^{-n+1} \int_0^t (\sinh \sqrt{-k} \rho)^{n-1} d\rho \qquad k<0$$

by $s \geq 0$, $s \leq \sinh s \leq s e^s \Rightarrow b_n t^n \leq V_k(t) \leq b_n t^n e^{(n-1)\sqrt{-k}\, t}$ $\quad k<0$

当$k \leq 0$时, $\dfrac{Vol_g(B_x(R))}{Vol_g(B_x(r))} \leq \dfrac{V_k(R)}{V_k(r)} = \left(\dfrac{R}{r}\right)^n e^{(n-1)\sqrt{-k}R}$

若 $Ric_{(M,g)}(x) \geq k g_x$，则有 $Vol_g(B_x(R)) \leq (\frac{R}{r})^n e^{\sqrt{(n-1)|k|}R} Vol_g(B_x(r))$ (*)

Obviously，当 $k>0$ 时，(*) 仍成立。

综上所述，只要 (M,g) 完备，$Ric_{(M,g)}(x) \geq k g_x$，$k \in R$，

便有 $Vol_g(B_x(R)) \leq (\frac{R}{r})^n e^{\sqrt{(n-1)|k|}R} Vol_g(B_x(r))$.

Definition. Let (M,g) be a Rie. manifold, we say that a family (Ω_k) of open subsets of M is a uniformly locally finite covering of M if the following holds: $\bigcup_k \Omega_k = M$ and there exists an integer N such that each point $x \in M$ has a neighborhood which intersects at most N of the Ω_k's.

Lemma (M^n,g) Rie. manifold. complete. $Ric_{(M,g)} \geq kg$ and let $\rho>0$ be given. There exists a sequence $\{x_i\}$ of points of M such that for any $r \geq \rho$:

(i) the family $\{B_{x_i}(r)\}$ is a uniformly locally finite covering of M, and there is an upper bound for N in terms of n, ρ, r and k

(ii) for any $i \neq j$, $B_{x_i}(\rho/2) \cap B_{x_j}(\rho/2) = \phi$.

proof. ① by the above remark, we have

$$\forall x \in M,\ 0<r<R \quad Vol_g(B_x(r)) \geq e^{-\sqrt{(n-1)|k|}R} (\frac{r}{R})^n Vol_g(B_x(R)) \quad (1)$$

② claim $\exists \{x_i\} \subset M$ s.t. $M = \bigcup_{i=1}^{\infty} B_{x_i}(\rho)$ and $\forall i \neq j$ $B_{x_i}(\rho) \cap B_{x_j}(\rho) = \phi$ (2)

Let $X_\rho = \{ \{x_i\}_{i\in I} : x_i \in M, \forall i \in I,\ I$ countable and $\forall i \neq j, d_g(x_i,x_j) \geq \rho \}$

$\{x_i\}_{i\in I}$ 中元素与 I 一一对应，至多可数

1° X_ρ is paritially ordered by inclusion （自反性 反对称性 传递）

2° 全序子集都有上界，in fact，设 A 为 $\mathcal{A}$ 中全体元素的并，要证 A 也为 X_ρ 中的元素。首先任取 $a, b \in A$ 则 $\exists \{x_i\}_{i\in I}$ $\{y_j\}_{j\in J}$ s.t. $a = x_{i_0}$, $b = y_{j_0}$ for some $i_0 \in I$, $j_0 \in J$. 由于 $\mathcal{A}$ 为全序子集，不妨设 $\{x_i\}_{i\in I} \subset \{y_j\}_{j\in J}$ 则 $a \in \{y_j\}_{j\in J}$ $\Rightarrow a = y_{j_1}$ for some $j_1 \in J$ 又由于 $\{y_j\}_{j\in J} \in X_\rho$，若 $j_0 \neq j_1$ 则 $B_{y_{j_0}}(\rho) \cap B_{y_{j_1}}(\rho) = \phi$

i.e. $B_a(\rho) \cap B_b(\rho) = \phi$ $\Rightarrow$ A 中任意 2 个不同点之间的距离 $\geq \rho$，$\Rightarrow \forall a, b \in A$, $a \neq b$ $B_a(\rho/2) \cap B_b(\rho/2) = \phi$ 由 M 的第二可数性，A 至多可数，为 A 中元素编号得到 $\{z_j\} \in X_\rho$ 从而证明了 A 在 X_ρ 中。

于是由 Zorn 引理，X_ρ contains a maximal element $\{x_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ satisfies (2).

说明 $M = \bigcup_i B_{x_i}(\rho)$ 若 $\exists x_0 \in M$ 但 $x_0 \notin \bigcup_i B_{x_i}(\rho) \Rightarrow d(x_0, x_i) \geq \rho$ $\forall i$

$\{x_i\}_{i\in I} \cup \{x_0\}$ 在 X_ρ 中包含 $\{x_i\}_{i\in I}$，与極大性矛盾

③ Let $\{x_i\}_{i\in I}$ be such that (2) is satisfied. For $r>0$ and $x\in M$ we define

$$I_r(x) = \{i : x \in B_{x_i}(r)\}$$

By (1) we get that for $r \geq \rho$

$$\begin{aligned} Vol_g(B_x(r)) &\geq \frac{1}{2^n} e^{-2\sqrt{(n-1)|K|}\,r} Vol_g(B_x(2r)) \\ &\geq \frac{1}{2^n} e^{-2\sqrt{(n-1)|K|}\,r} \sum_{i\in I_r(x)} Vol_g(B_{x_i}(\rho/2)) \\ &\geq \frac{1}{2^n} e^{-\sqrt{(n-1)|K|}\,r}\, card\, I_r(x)\, e^{-2\sqrt{(n-1)|K|}\,r} \left(\frac{\rho}{4r}\right)^n Vol_g(B_{x_i}(2r)) \\ &\geq \cdots Vol_g(B_x(r)). \end{aligned}$$

Now, let $B_{x_i}(r)$ be given, $r\geq\rho$, and suppose that N balls $B_{x_j}(r)$ have a nonempty intersection with $B_{x_i}(r)$, $j\neq i$. Then, obviously, $card\, I_{2r}(x_i) \geq N+1$. Hence, $N \leq C(n,\rho,2r,K)-1$ and this proves the Lemma #

Sobolev spaces

Let (M,g) be a Riemannian manifold. For k an integer and $u\in C^\infty(M)$, $\nabla^k u$ denotes the kth covariant derivative of u.

$$|\nabla^k u|^2 = g^{i_1j_1}\cdots g^{i_kj_k} (\nabla^k u)_{i_1\cdots i_k} (\nabla^k u)_{j_1\cdots j_k}$$

For k an integer and $p\geq 1$ real, we denote by $C_k^p(M)$ the space of smooth functions $u\in C^\infty(M)$ such that $|\nabla^j u|\in L^p(M)$ for any $j=0,\dots,k$.

Hence, $C_k^p(M) = \{u\in C^\infty(M) : \int_M |\nabla^j u|^p dv(g) < \infty \ \forall j=0,\dots,k\}$

If M is compact, one has that $C_k^p(M) = C^\infty(M)$ for all k and $p\geq 1$.

Definition 2.1: The Sobolev space $H_k^p(M)$ is the completion of $C_k^p(M)$ with respect to the norm

$$\|u\|_{H_k^p} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p dv(g)\right)^{1/p}$$

Noting that a Cauchy sequence in $C_k^p(M)$ is also a Cauchy sequence in $L^p(M)$, and that a Cauchy sequence in $C_k^p(M)$ which converges to 0 in $L^p(M)$ converges to 0 in $C_k^p(M)$, the Sobolev spaces $H_k^p(M)$ can be seen as subspaces of $L^p(M)$. This is the point of vue we adopt in the sequence. More precisely, one can look at $H_k^p(M)$ as the space of functions u in $L^p(M)$ which are limit in $L^p(M)$ of a Cauchy sequence (u_m) in $C_k^p(M)$.

$\vec{L}^p(M)$ $\qquad$ (M^n, g) Rie. manifold

Def 1: X rough vector field i.e $X: M \to TM$ $\quad X(p) \in T_pM$ 不谈X的连续性, 光滑性.

if for each local chart $(U, \varphi; x^i)$, $X|_U = X^i \frac{\partial}{\partial x^i}$ $\quad X^i \circ \varphi^{-1}$ is a measurable function on $\mathbb{R}^n$, $\forall i = 1, \cdots, n$

then we say that X is measurable on M.

Remark: if X is measurable on M, then $|X|^p$ is a measurable function on M.

Hence we define

$p \geq 1$. $\quad \vec{L}^p(M) = \{ X \text{ is a measurable rough vector field: } (\int_M |X|^p dv(g))^{1/p} < \infty \}$.

claim: $\vec{L}^p(M)$ is Banach space.

proof. 1. Assume that $\{X_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\vec{L}^p(M)$ and Ω is precompact subset of M.

Since Ω is precompact, Ω can be covered by a finite number of charts $\{U_m, \chi_m\}_{m=1}^{N}$

$\exists C > 0 \quad C^{-1}\delta_{ij} \leq g_{ij}^m \leq C\delta_{ij}$ as bilinear forms $\forall m = 1, 2, \cdots, N$

and Let $\{\eta_m\}$ be a smooth partition of unity subordinate to the covering $\{U_m\}$.

Fixed m, $\{\eta_m X_n\}$ is a Cauchy sequence in $\vec{L}^p(U_m)$ and $L^p(\chi_m(U_m)) \times \cdots \times L^p(\chi_m(U_m))$ is complete.

there exist $X^{(m)} \in \vec{L}^p(U_m)$ such that $\eta_m X_n \to X^{(m)}$ in $\vec{L}^p(U_m)$.

More precisely, $\quad \eta_m X_n = \eta_m X_n^i \frac{\partial}{\partial x^i}$

$\forall i, \ \exists X^{(m)i} \quad (\eta_m X_n^i) \circ \chi_m^{-1} \to X^{(m)i} \circ \chi_m^{-1}$ in $L^p(\chi_m(U_m))$

i.e $\int_{U_m} |(\eta_m X_n^i) \circ \chi_m^{-1} - X^{(m)i} \circ \chi_m^{-1}|^p dx \to 0$ as $n \to \infty$

Set $X^{(m)} = X^{(m)i} \frac{\partial}{\partial x^i}$

$$\Rightarrow \left(\int_{U_m} |\eta_m X_n - X^{(m)}|^p dv(g)\right)^{1/p} = \left(\int_{U_m} |(\eta_m X_n^i - X^{(m)i})(\eta_m^j X_n - X^{(m)j}) g_{ij}|^p \sqrt{\det(g_{ij}^m)}\, dx\right)^{1/p}$$

$$\leq C \cdot C^{\frac{n}{2p}} \left(\int_{U_m} \left(\Sigma_i (\eta_m X_n^i - X^{(m)i})^2\right)^{\frac{p}{2}} dx\right)^{1/p}$$

$$\leq C \cdot C^{\frac{n}{2p}} \left(\int_{\chi_m(U_m)} \left[\Sigma_i ((\eta_m X_n^i) \circ \chi_m^{-1} - X^{(m)i} \circ \chi_m^{-1})^2\right]^{\frac{p}{2}} dx\right)^{1/p} \to 0 \quad \text{as } n \to \infty$$

Set $X = \sum_{m=1}^{N} X^{(m)}$ well-defined.  相交处 a.e. 相等

$$\|X\|_{\vec{L}^p(\Omega)} = \|\sum_{m=1}^{N} X^{(m)}\|_{\vec{L}^p(\Omega)} \leq \sum_{m=1}^{N} \|X^{(m)}\|_{\vec{L}^p(U_m)} < \infty$$

and $\|X_n - X\|_{\vec{L}^p(\Omega)} = \|\sum_{m=1}^{N} \eta_m X_n - X\|_{\vec{L}^p(\Omega)} \leq \Sigma_{m=1}^{N} \|\eta_m X_n - X^{(m)}\|_{\vec{L}^p(U_m)} \to 0$ as $n \to \infty$.

therefore $X \in \vec{L}^p(\Omega)$ and $X_n \to X$ in $\vec{L}^p(\Omega)$.

2. Assume that $\{X_n\}$ is a Cauchy sequence in $\vec{L}^p(M)$, and Let $\{\Omega_i\}_{i=1}^{\infty}$ is a precompact exhaustion of M, i.e. Ω_i is precompact and $\Omega_i \subset \Omega_{i+1}$ for all i and $\bigcup_{i=1}^{\infty} \Omega_i = M$.

Choose a family smooth functions with compact support $\{\eta_i\}$ satisfying

$\eta_i = 1$ on Ω_i. $\quad$ spt $\eta_i \subset \Omega_{i+1}$. 

Fixed i, $\{\eta_i X_n\}$ is a Cauchy sequence in $\vec{L}^p(\Omega_{i+1})$. by 1. there exist $X^{(i)} \in \vec{L}^p(\Omega_{i+1})$ s.t $\eta_i X_n \to X^{(i)}$ in $\vec{L}^p(\Omega_{i+1})$

Set $X = X^i$ in Ω_i well-defined (|||) 一致

$$\int_M |X|^p \, dv(g) = \int_M |\lim_i \eta_i X|^p \, dv(g) = \int_M \lim_i \eta_i^p |X|^p \, dv(g) \underset{\substack{\uparrow \\ \text{monotone} \\ \text{convergence} \\ \text{theorem}}}{=} \lim_i \int_M \eta_i^p |X|^p \, dv(g) \leq \int_M |X|^p \, dv(g) < \infty$$

Given $\varepsilon > 0$, since $\{X_n\}$ is a Cauchy sequence, then there exists a increasing sequence $\{n_k\}$ such that $\| X_{n_k} - X_{n_{k+1}} \|_{\vec{L}^p(M)} < \frac{\varepsilon}{2^k}$.

It is easy to see that there exists a increasing sequence such that $\| \eta_{m_k} X_{n_k} - X_{n_k} \| < \frac{\varepsilon}{2^k}$

Hence
$$\begin{aligned}
\| X - X_{n_k} \|_{\vec{L}^p(M)} \leq\ & \underbrace{\| X - \eta_{m_j} X \|_{\vec{L}^p(M)}}_{\to 0} + \| \eta_{m_j} X - X^{(m_k)} \|_{\vec{L}^p(M)} \to \| X - X^{(m_k)} \| \to 0 \quad k \gg 1. \\
& + \| X^{(m_k)} - \eta_{m_k} X_{n_j} \|_{\vec{L}^p(M)} \to \| X^{(m_k)} - \eta_{m_k} X \| \\
& + \| \eta_{m_k} X_{n_j} - \eta_{m_k} X_{n_k} \|_{\vec{L}^p(M)} \leq \| X_{n_j} - X_{n_k} \| \leq \frac{\varepsilon}{2^k} + \cdots + \frac{\varepsilon}{2^{j-1}} = \varepsilon \cdot \frac{1}{2^{k-1}} \\
& + \| X_{n_k} - \eta_{m_k} X_{n_k} \|_{\vec{L}^p(M)} < \frac{\varepsilon}{2^k}.
\end{aligned}$$

for $j > k$. as $j \to \infty$.

$\Rightarrow$ $X_{n_k} \to X$ in $\vec{L}^p(M)$

$+$ $\{X_n\}$ is a Cauchy sequence in $\vec{L}^p(M)$

$\Rightarrow$ $X_{n_k} \to X$ in $\vec{L}^p(M)$.

$H_1^p(M)$: completion of $C_1^p(M)$

$u \in H_1^p(M)$ i.e. there exist a Cauchy sequence $\{u_m\}$ in $H_1^p(M)$

s.t. $u = \lim\limits_m u_m$ in L^2

由于 $\{u_m\}$ Cauchy sequence. 则 $\int_M |\nabla u_m - \nabla u_n|^p \to 0$ as $n, m \to \infty$

i.e ∇u_m is a Cauchy sequence in $\vec{L}^p(M)$

by the completeness of $\vec{L}^p(M)$, $\exists$ $X \in \vec{L}^p(M)$. s.t. $\nabla u_m \to X$ in $\vec{L}^p(M)$

想法 $\vec{D}(M)$: M上具有紧支集的向量场的全体

$$\int_M \langle \nabla u_m, \chi \rangle = -\int_M u_m \cdot \operatorname{div} \chi \qquad \forall \chi \in \vec{D}(M)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_M \langle X, \chi \rangle \qquad -\int_M u \operatorname{div} \chi$$

$$\Rightarrow \int_M \langle X, \chi \rangle = -\int_M u \operatorname{div} \chi \quad (*)$$

$\Rightarrow$ define $\nabla u := X$ 由 $(*)$ 式唯一确定.

i.e we have

$$\int_M \langle \nabla u, \chi \rangle = -\int_M u \operatorname{div} \chi \quad \forall \chi \in \vec{D}(M)$$

说明 完备化得到的 Sobolev 空间中的函数是拥有我们期待的弱导数的.

相似地. 我们可以对高阶张量定义 $\vec{L}^p(M)$. test spaces and weak 导数:

Remark: 上述 Sobolev 空间的定义当然是没问题的. 不借助分布的语言可以将该定义的良定性严格说清楚吗?

Alexander Grigor'yan 用分布的语言定义了 $H^p(M)$. $H^p_0(M)$. 写得很清楚可参考。[Heat Kernel and Analysis on Manifolds] 有中文版.

prop. If M is compact $H^p_k(M)$ does not depend on the Riemannian metric.

proof. (M,g). $(M,\tilde g)$ two Rie. manifolds. M compact

取有限坐标覆盖 $\{\Omega_m, \phi_m\}_{m=1}^N$ 满足下面的性质:

$$C^{-1}\tilde g^m_{ij} \le g^m_{ij} \le C\tilde g^m_{ij}, \ \forall m=1,\cdots,N \quad \text{as bilinear} \quad \text{where } C\geq 1$$

Let $\{\eta_m\}$ 为从属于 $\{\Omega_m\}$ 的单位分解. 取 $u\in H^p_k(M,\tilde g)$

$$\|\eta_m u\|_{H^p_k(M,g)} = \sum_{j=0}^{N}\left(\int_M |\nabla^j(\eta_m u)|^p dv(g)\right)^{1/p}$$
$$= \sum_{j=0}^{N}\left(\int_{\phi_m(\Omega_m)} |\nabla^j(\eta_m u)\circ\phi_m^{-1}|^p \sqrt{\det(g^m_{ij})}\, dx\right)^{1/p}$$
$$\le C^{\frac{n}{2p}} \sum_{j=0}^{N}\left(\int_{\phi_m(\Omega_m)} |\nabla^j(\eta_m u)\circ\phi_m^{-1}|^p \sqrt{\det(\tilde g^m_{ij})}\, dx\right)^{1/p}$$
$$= C^{\frac{n}{2p}} \|\eta_m u\|_{H^p_k(M,\tilde g)}$$

Set $u=\sum_{j=1}^N \eta_j u$

$$\|u\|_{H^p_k(M,g)} \le \sum_{m=1}^N \|\eta_m u\|_{H^p_k(M,g)}$$
$$\le \sum_{m=1}^N C^{\frac{n}{2p}} \|\eta_m u\|_{H^p_k(M,\tilde g)}$$
$$\le C^{\frac{n}{2p}} N \|u\|_{H^p_k(M,\tilde g)} < \infty$$

#

Remark: prop is not any more true for non-compact manifolds.

Lemma Let (M,g) be a Riemannian manifold and $u: M\to R$ a Lipschitzian function on M which equals zero outside a compact subset of M. Then $u\in H^p_1(M)$ for any $p\geq 1$.

proof. Let $u: M\to R$ be a Lipschitzian function on M which equals zero outside a compact subset K of M. Let $(\Omega_k,\phi_k)_{k=1,\cdots,N}$ be a finite number of charts such that $K\subset\bigcup_{k=1}^N\Omega_k$ and such that for any $k=1,\cdots,N$

$$\phi_k(\Omega_k) = B^e_0(1) \text{ and } R^{-1}\delta_{ij}\le g^k_{ij}\le R\delta_{ij} \quad \text{as bilinear forms} \quad R\geq 1.$$

where $B^e_0(1)$ is the unit euclidean ball of center 0 in R^n, and the g^k_{ij}'s are the components of g in (Ω_k,ϕ_k).

Let $(\alpha_k)_{k=1,\cdots,N}$ be a smooth partition of unity subordinate to the covering

$(\Omega_k)_{k=1,\cdots,N}$. One easily gets that for any $k=1,\cdots,N$, $(\alpha_k u)\circ\phi_k^{-1}$ is a lipschitzian function on $B_0^e(1)$ $\Rightarrow$ $(\alpha_k u)\circ\phi_k^{-1}\in H_1^p(B_0^e(1))$ for $p\geq 1$

$$\Rightarrow \alpha_k u\in H_1^p(M)$$

$u=\sum_{k=1}^{N}\alpha_k u \Rightarrow \|u\|_{H_1^p(M)}\leq\sum_{k=1}^{N}\|\alpha_k u\|_{H_1^p(M)}<\infty$ #

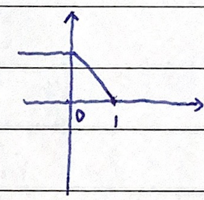
Remark: if $u:M\to R$ a lipschitzian function on M which equals zero outside a compact subset of M, then $u\in \mathring{H}_1^p(M)$ for any $p\geq 1$.

Just note that if $\{u_m\}\in C_1^p(M)$ converges to u in $H_1^p(M)$, and take $\alpha\in C_0^\infty(M):=D(M)$ with $\alpha=1$ on $\operatorname{spt} u$, then $\{\alpha u_m\}$ converges to $\alpha u=u$ in $H_1^p(M)$.

Theorem If (M,g) is complete, then for all $p\geq 1$, $\mathring{H}_1^p(M)=H_1^p(M)$

proof. Let $f:R\to R$ be defined by

$$f(t)=\begin{cases}1, & t\leq 0\\ 1-t, & 0\leq t\leq 1\\ 0, & t\geq 1\end{cases}$$



想证 $H_1^p(M)\subset\mathring{H}_1^p(M)$. 由于 $\mathring{H}_1^p(M)$是完备的. $H_1^p(M)$是 $C_1^p(M)$的完备化

故只需证 $C_1^p(M)\subset\mathring{H}_1^p(M)$. 由于 $\mathring{H}_1^p(M)$为 $H_1^p(M)$的闭子空间

从而只需证 $C_1^p(M)$中每个元素可以用 $\mathring{H}_1^p(M)$中序列在 $\|\cdot\|_{H_1^p(M)}$范数下逼近.

Let $u\in C_1^p(M)$ where $p\geq 1$ is some given real number.

Let x be some point of M and set

$$u_j(y)=u(y)\,f(d_g(x,y)-j)=\begin{cases}u(y) & d_g(x,y)\leq j\\ u(y)(1-d_g(x,y)+j), & j\leq d_g(x,y)\leq j+1\\ 0 & d_g(x,y)\geq j+1\end{cases}$$

$\Rightarrow u_j$ is a Lipschitzian function which equals zero outside a compact subset of M.

$\Rightarrow u_j\in\mathring{H}_1^p(M)$

Independently, one clearly has that for any j,

$$\left(\int_M|u_j-u|^p\,dv(g)\right)^{1/p}\leq\left(\int_{M\setminus B_x(j)}|u|^p\,dv(g)\right)^{1/p}$$

$$\left(\int_M|\nabla(u_j-u)|^p\,dv(g)\right)^{1/p}\leq\left(\int_{M\setminus B_x(j)}|\nabla(u_j-u)|^p\,dv(g)\right)^{1/p}$$

$$\leq\left(\int_{M\setminus B_x(j)}|\nabla u|^p\,dv(g)\right)^{1/p}+\left(\int_{M\setminus B_x(j)}|u|^p\,dv(g)\right)^{1/p}$$

in fact, on $B_x(j+1)^c$ $u_j\equiv 0$

on $B_x(j+1)\setminus B_x(j)$ $|\nabla(u-u_j)|=|\nabla(u(d_g(x,y)-j))|=|\nabla u\cdot(d_g(x,y)-j)+u\,\nabla(d_g(x,y))|$

$$\leq|\nabla u|+|u|$$

令 $j\to+\infty$. ✓ #